18·04·2023

Ordinary Differential Equations and Vector Calculus

UNIT - I : First Order (ODE) Ordinary differential Eq.

- Exact differential equations
- Equations reducible to exact differential equations.
- Linear equations
- Bernoulli's equations
- Orthogonal Trajectories (only in Cartesian Coordinates)
- Applications:
  - Newton's law of cooling
  - Law of natural growth and decay.

UNIT - II : Ordinary Differential Equations (ODEs) of Higher order

- Second Order linear differential equations with constant coefficients
- Non-homogenous terms of the type:
  - e^{ax}
  - $\sin ax$
  - $\cos ax$
  - polynomial of x
  - $e^{ax} \cdot V(x)$ and $V(x) \cdot x$
- Method of variation of parameters

· Equations reduced to linear (ODE) Ordinary Differential Equation with constant coefficient:

· Legendre's equation.
· Cauchy - Euler equation

· Application: Electric circuit

UNIT - III: Laplace Transforms

· Laplace transform of standard functions
· First shifting theorem.
· Second shifting theorem
· Unit step function
· Dirac delta function
· Laplace transforms of functions when they are multiplied and divided by 't'.
· Laplace transforms of derivatives and integral of function.
· Evaluation of integrals by Laplace transforms
· Laplace transform of periodic functions.
· Inverse Laplace transform by different different methods
· Convolution theorem (without proof)
· Applications: Solving value problems by Laplace Transform method.

UNIT - IV : Vector Differentiation

- Vector point functions and scalar point functions
- Gradient
- Divergence and Curl
- Directional derivatives
- Tangent plane and normal line
- Vector identities
- Scalar potential functions
- Solenoidal and Irrotational vectors.

UNIT - V : Vector Integration

- Line integrals
- Surface integrals
- Volume integrals
- Theorems of Green, Gauss and Stokes (without proofs) and their applications.

Differentiation Formulas

- $\frac{d}{dx} x^n = n x^{n-1}$
- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \log x = \frac{1}{x}$
- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$

- $\frac{d}{dx} u.v = uv' + vu'$
- $\frac{d}{dx} \left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
- $\frac{d}{dx} f(g(x)) = fg' + gf'$
- $\frac{d}{dx} k.f(x) = k.f'(x)$

- $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}$
- $\frac{d}{dx} \cot^{-1} x = \frac{-1}{x^2+1}$
- $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$
- $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$

- $\frac{d}{dx} a^x = a^x \log a$
- $\frac{d}{dx} \log_a x = \frac{1}{x \log a}$
- $\frac{d}{dx} e^{fx} = e^{fx} . f'x$

12.04.2023

UNIT - I

First Order Differential Equation (ODE)

Differential Equation:

An equation involving derivatives with respective to one (1) or more independent variables is called a Differential equation.

Example: $$\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + y = 3$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y} = 0$$

Types of Differential Equation:

There are two types of Differential Equations:

- Ordinary differential equation [ODE]
- Partial differential equation [PDE]

Ordinary Differential Equation:

An equation involving the derivatives with respect to one independent variable is called Ordinary differential equation.

Example: $\frac{d^2y}{dx^2} + 2x\frac{dy}{dz} + y = 3$

$$\frac{d^2x}{dy^2} + (2y-1)\frac{dx}{dy} + (y+3)x = \tan 4y$$

- Partial Differential Equation

An equation involving the derivatives with respect to two or more independent variables is called Partial Differential equation.

Example: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c}\frac{\partial^2 u}{\partial y^2}$$

- Order of Differential Equation

The order of Differential Equation is the highest derivative appearing in the given differential equation.

Example: $\left(\frac{d^2y}{dx^2}\right)^4 + \frac{dy}{dx} + 3y = 0$

Order $= 2$ Degree $= 4$.

Degree of Differential Equation

The power of the highest derivative in the differential equation is called Degree of differential equation. and Differential equation has been made free from radicals and fractions as per the derivatives are consider.

18) $$y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$y - x\frac{dy}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(y - x\frac{dy}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$y^2 - 1 - 2xy\left(\frac{dy}{dx}\right) + x^2\left(\frac{dy}{dx}\right)^2 - \left(\frac{dy}{dx}\right)^2 = 0$$

$$\left(\frac{dy}{dx}\right)^2(x^2-1) - 2yx\frac{dy}{dx} + (y^2-1) = 0$$

$\therefore$ Order $= 1$

Degree $= 2$.

First Order differential equation and first degree differential equation

An equation of the form $\frac{dy}{dx} = f(x, y)$ is called the differential equation of first order and first degree.

- First order differential equation can be ~~casse~~ classified as follows:
- Variable separable Method
- Homogenous and Non-Homogenous
- Exact and non-exact equation
- Liner equation
- Bernoulis equation.

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(1) Exact Differential Equation · 1M ·

Let $M(x, y)\,dx + N(x, y)\,dy = 0$ be the first order and first degree Differential equation where M and N are real valid functions for some (x, y).

$Mdx + Mdy$ is exact then there exists a function $f(x, y)$ ~~thus~~ ~~exact~~ such that

$M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$ thus exact differential equation can always derived from general equation by differentiating without any subsiquent multiplication (or)

The differential equation of $Mdx + Ndy = 0$ is said to be exact differentiable equation if it satisfies the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

- Working Rules to solve the exact Differential equation.
- The equation should be in the form of $Mdx + Mdy = 0$.
- Test for the exactness by $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
- If the differential equation is exact then integral terms in 'M' with respect to x keeping 'y' as constand and integral terms in 'N' which are free from 'x' with respect to 'y'.
- General equation

$$\boxed{[\text{or}]\ \text{General solution} = \int_{y\ \text{const}} M dx + \int_{(\text{without } x \text{ terms})} N dy = c}$$

15.04.23

2Q) Solve the differential equation

$$(2x - y + 1)dx + (2y - x - 1)dy = 0$$

Solution:-

$$(2x - y + 1)dx + (2y - x - 1)dy = 0 \quad \text{--- (1)}$$

(1) is in the form of $M dx + N dy = 0$

where,

$$M = 2x - y + 1 \qquad N = 2y - x - 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2x - y + 1) \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial y}(2y - x - 1)$$

$$= \cancel{2\left(\frac{x^2}{2}\right) - yx + x} \qquad = \cancel{2\frac{y^2}{2} - xy - y}$$

$$= 0 - 1 + 0 \qquad = 0 - 1 - 0$$

$$= -1 \qquad = -1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

∴ (1) is exact

General Solution of (1) is

$$\underset{(y \text{ constant})}{\int M dx} + \underset{(\text{eleminant } x \text{ terms})}{\int N dy} = c$$

$$\int (2x - y + 1) dx + \int (2y - x - 1) dy = c$$

3Q

$$2\left(\frac{x^2}{2}\right) - yx + x + \int (2y-1)\,dy = c$$

$$x^2 - xy + x + 2\frac{y^2}{2} - y = c$$

$$x^2 + y^2 - xy + x - y = c$$

$\therefore x^2 + y^2 - xy + x - y = c$ is the General solution for $(2x-y+1)\,dx + (2y-x-1)\,dy = 0$.

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39 Solve $(x^2 + 2\sin y)\,dx + (2x\cos y + y)\,dy = 0$

Solution:-

$$(x^2 + 2\sin y)\,dx + (2x\cos y + y)\,dy = 0 \quad — (1)$$

Eq (1) is in the form of ~~Mdy~~ $Mdx + Ndy = 0$

where, $M = x^2 + 2\sin y$ $\quad N = 2x\cos y + y$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2\sin y) \qquad \frac{\partial N}{\partial x} = 2\cos y\,(1) + 0$$

$$= 0 + 2(\cos y) \qquad = 2\cos y$$

$$= 2\cos y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

According to Exact equation,

General Solution for Eq (1) is

$$\underset{y\ \text{constant}}{\int M\,dx} + \underset{\substack{\text{deminant}\\ x\ \text{term}}}{\int N\,dy} = c$$

$$\int (x^2 + 2\sin y)\,dx + \int y\,dy = c$$

$$\frac{x^3}{3} + 2\sin y . x + \frac{y^2}{2} = c$$

$$\frac{x^3}{3} + \frac{y^2}{2} + 2x\sin y = c$$

$2x^3 + 3y^2 + 12x\sin y = 6c$ is the general solution

4Q Solve $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$

Solution :-

$$(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$$

$$(y^2 - 2xy)\,dx - (x^2 - 2xy)\,dy = 0 \quad \text{—— } (1)$$

Eq (1) is in the form $M\,dx + N\,dy = 0$

where, $M = y^2 - 2xy$ $\qquad N = -(x^2 - 2xy)$

$$= -x^2 + 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^2 - 2xy) \qquad \frac{\partial N}{\partial x} = -2x + 2y$$

$$= 2y - 2x \qquad\qquad = 2y - \cancel{2y}\ 2x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq① is exact

General Solution for Eq① is

$$\int_{y\ \text{const}} M\,dx + \int N\,dy = c$$

y const   eleminate x term

$$\int (y^2 - 2xy)\,dx + \int (-x^2 + 2xy)\,dy = c$$

$$\Rightarrow xy^2 - \frac{\cancel{2}x^2y}{\cancel{2}} - \cancel{\frac{x^3}{3}} = c$$

$$\Rightarrow xy^2 - x^2y - \cancel{\frac{x^3}{3}} = c$$

$\therefore\ xy^2 - x^2y = c$ is the general solution for $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$.

Solve $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Solution:

$$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$$

$$\Rightarrow (\sin x + x\cos y + x)\,dx + (y\cos x + \sin y + y)\,dy = 0$$

— ①

① is in form $Mdx + Ndy = 0$

where,

$M = y\cos x + \sin y + y \quad N = \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y\cos x + \sin y + y) \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\sin x + x\cos y + x)$$

$$\frac{\partial M}{\partial y} = \cos x + \cancel{}\cos y + 1 \qquad \frac{\partial N}{\partial x} = \cos x + \cancel{}\cos y + 1$$

$$= \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq ① is exact.

General solution for Eq ① is

$$\underset{(y\ \text{const})}{\int M\,dx} + \underset{(\text{eleminat } x \text{ tom})}{\int N\,dy} = c$$

$$\Rightarrow \underset{y\ \text{const}}{\int (y\cos x + \sin y + y)\,dx} + \int (\sin x + x\cos y + x)\,dy = c$$

$$\Rightarrow y(+\sin x) + \sin y \cdot x + y \cdot x + 0 = c$$

$\therefore$ $y\sin x + x\sin y + xy = c$ is the general solution of the equation $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + y} = 0$

69 (H.W) Solve $2xy\,dy - (x^2 - y^2 + 1)\,dx = 0$

Solution :-

$$-(x^2 - y^2 + 1)\,dx + 2xy\,dy = 0 \quad \text{--- (1)}$$

Eq (1) is in form $M\,dx + N\,dy = 0$

where,

$M = -x^2 + y^2 - 1 \qquad N = 2xy$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x^2 + y^2 - 1) \qquad \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy)$$

$$= \cancel{-x^2}\ 2y \qquad\qquad = 2y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq (1) is exact

General solution of Eq (1) is

$$\underset{(y\ \text{const})}{\int M\,dx} + \underset{(\text{eleminate } x \text{ term})}{\int N\,dy} = c$$

$$\Rightarrow \int (-x^2 + y^2 - 1)\,dx + \int 0\,dy = c$$

$$\Rightarrow -\frac{x^3}{3} + y^2 x - x = c$$

$\therefore\ -\dfrac{x^3}{3} + y^2 x - x = c$ is the general solution of the equation $2xy\,dy - (x^2 - y^2 + 1)\,dx = 0.$

18.04.2023

- Non Exact (reducible to Exact)

If the equation is not exact. But sometimes we may make it exact by multiplying with a suitable factor is called an integrating factor.
Multiplying the given equation with an integrating factor, the equation becomes exact.

2 Method 1: Integrating factor for a homogenous equation:

If the given differential equation is in the form

$$M dx + N dy = 0$$ is non exact and homogenous equation and also $Mx + Ny \neq 0$.

Then integrating factor $I_F = \frac{1}{Mx + Ny}$

74 Solve $x^2 y\, dx - (x^3 + y^3) dy = 0$

Solution:

$$x^2 y\, dx - (x^3 + y^3) dy = 0 \quad \text{--- (1)}$$

Eq (1) is in form $M dx + N dy = 0$

where,

$M = x^2 y$

$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2 y)$

$= x^2(1)$

$= x^2$

$N = -x^3 - y^3$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-x^3 - y^3)$

$= -3x^2 - (0)$

$= -3x^2$

$\therefore$ (1) is non Exact.

Eq ① is Homogenous (Since powers are equal) (of degree 3)

Integrating factor $I_F = \dfrac{1}{Mx + Ny}$

$$= \frac{1}{(x^2y)x + -(x^3+y^3)y}$$

$$= \frac{1}{\cancel{x^3y} - \cancel{x^3y} + (-y^4)}$$

$$I_F = \frac{-1}{y^4} \qquad \left(\because \frac{d}{dx}(x^n) = nx^{n-1}\right)$$

Multiplying I_F in Eq ①:-

$$\Rightarrow \frac{-x^2y}{y^4}\,dx - \left(\frac{x^3+y^3}{y^4}(-1)\right)dy = 0$$

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0 \quad \text{—— ②}$$

Eq ② is in form of $Mdx + Ndy = 0$

where, $M_1 = \dfrac{-x^2}{y^3}$

$$\frac{\partial M_1}{\partial y} = -x^2\left(\frac{-3}{y^4}\right) = \frac{3x^2}{y^4}$$

$N_1 = \dfrac{x^3}{y^4} + \dfrac{1}{y}$

$$\frac{\partial N_1}{\partial x} = \frac{1}{y^4}(3x^2) + \frac{1}{y}(0) = \frac{3x^2}{y^4}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$ Eq ② is exact

General solution of Eq ② is

$$\int_{y\ \text{const}} M_1 dx + \int_{x\ \text{eleminat}} N_1 dy = c \qquad \left| \because \int x^n dx = \frac{x^{n+1}}{n+1} \right)$$

$$\int \left(\frac{-x^2}{y^3} \right) dx + \int \frac{1}{y} dy = c$$

$$\frac{-1}{y^3} \int x^2 dx + \log |y| = c$$

$\therefore \quad \dfrac{-x^3}{3y^3} + \log y = c$ is the general solution for the equation $x^2y - (x^3+y^3)\,dy = 0$.

Q. Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ —— ①

Solution :-

Eq ① is in form $Mdx + Ndy = 0$

where,

$M = x^2y - 2xy^2 \qquad N = -x^3 + 3x^2y$

$$\frac{\partial M}{\partial y} = x^2(1) - 2x(2y) \qquad \frac{\partial N}{\partial x} = \cancel{x^3(0)+}$$

$$= -3x^2 + 3y(2x)$$

$$= x^2 - 4xy \qquad = -3x^2 + 6xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial y}$$

Eq ① is non exact.

Eq ① is homogenous of degree 3.

Integrating factor $I_F = \dfrac{1}{Mx + Ny}$

$$= \frac{1}{(x^2y - 2xy^2)x + (-x^3 + 3x^2y)y}$$

$$= \frac{1}{\cancel{x^3y} - 2x^2y^2 - \cancel{x^3y} + 3x^2y^2}$$

$$I_F = \frac{1}{x^2y^2}$$

Multiplying I_F with Eq ①

$$\frac{x^2y - 2xy^2}{x^2y^2}\,dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right)dy = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \quad \text{— ②}$$

Eq ② is in form $M_1dx + N_1dy = 0$

where $M_1 = \dfrac{1}{y} - \dfrac{2}{x}$ $\qquad N_1 = \dfrac{-x}{y^2} + \dfrac{3}{y}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{-y^2} - \frac{2}{x}(0) \qquad \frac{\partial N_1}{\partial x} = \frac{-1}{y^2}(1) + \frac{3}{y}(0)$$

$$= \frac{-1}{y^2} \qquad\qquad = \frac{-1}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Eq ② is Exact.

General solution of Eq ② is

$$\int_{y\ \text{const}} M\,dx + \int_{x\ \text{const}} N\,dy = c$$

$$\int \frac{3}{y}\,dy$$

$$\Rightarrow \int \left(\frac{1}{y} - \frac{2}{x}\right)dx + \cancel{\int \frac{3}{y}\,dy} = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$\therefore$ $\frac{x}{y} - 2\log x + 3\log y = c$ is the general solution of the equation $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)\,dy = 0$

9Q H.W Solve $y^2dx + (x^2 - xy - y^2)\,dy = 0$

10Q H.W Solve $(x^2 + 2y^2)\,dx + xy\,dy = 0$.

9 Solution: $y^2dx + (x^2 - xy - y^2)\,dy = 0$ — ①

Eq ① is in form $Mdx + Ndy = 0$

where,

$M = y^2$ $\qquad$ $N = x^2 - xy - y^2$

$\frac{\partial M}{\partial y} = 2y$ $\qquad$ $\frac{\partial N}{\partial x} = 2x - y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq ① is not exact

Eq ① is homogenous. (degree two).

Integrating factor $I_F = \dfrac{1}{Mx + Ny}$

$$I_F = \frac{1}{y^2x + (x^2 - xy - y^2)y}$$

$$= \frac{1}{\cancel{y^2x} + x^2y - \cancel{y^2x} - y^3}$$

$$I_F = \frac{1}{x^2y - y^3}$$

Multiplying I_F with Eq 1:

$$\frac{y^2\,dx + (x^2 - xy - y^2)\,dy}{x^2y - y^3} = 0$$

$$\frac{y}{x^2 - y^2}\,dx + \frac{x^2 - xy - y^2}{x^2y - y^3}\,dy = 0 \quad -2$$

Eq 2 is in form $M dx + N dy = 0$

$M = \dfrac{y}{x^2 - y^2}$ $\qquad$ $N = \dfrac{x^2 - xy - y^2}{x^2y - y^3}$

$$\frac{\partial M}{\partial y} = \frac{(x^2 - y)(1) - y(-1)}{(x^2 - y^2)^2}$$

$$= \frac{x^2 \cancel{- y} \cancel{+ y}}{(x^2 - y)^2}$$

$$= \frac{x^2}{(x^2 - y^2)^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2y - y^2)(2x - y) - (x^2 - xy - y^2)(2x)}{(x^2y - y^3)^2}$$

$$= \frac{2x^3y - \cancel{2xy^2} - x^2y^2 + y^3 - 2x^3 + x^2y + \cancel{2xy^2}}{(x^2 \cancel{y} - y^2)^2\,y^2}$$

$$= \frac{x^2}{(x^2 - y^2)}$$

10 Solution: $(x^2+2y^2)dx + xy\,dy = 0$ — ①

Eq ① is in form $Mdx + Ndy = 0$

where,

$M = x^2+2y^2$ $\qquad$ $N = xy$

$\dfrac{\partial M}{\partial y} = 4y$ $\qquad$ $\dfrac{\partial N}{\partial x} = y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Integrating factor $I_F = \dfrac{1}{Mx + Ny}$

$$I_F = \frac{1}{(x^2+2y^2)x + (xy)y}$$

$$= \frac{1}{x^3+2xy^2+xy^2}$$

$$I_F = \frac{1}{x^3+3xy^2}$$

Multiplying I_F with Eq①:

$$\frac{x^2+2y^2}{x^3+3xy^2}\,dx + \frac{xy}{x^3+3xy^2}\,dy = 0 \quad — ②$$

Eq② is in form $Mdx + Ndy = 0$

$M = \dfrac{x^2+2y^2}{x^3+3xy^2}$ $\qquad$ $N = \dfrac{xy}{x^3+3xy^2}$

$\dfrac{\partial M}{\partial y} = \dfrac{(x^3+3xy^2)(4y) - (x^2+2y^2)(6xy)}{(x^3+3xy^2)^2}$ $\qquad$ $\dfrac{\partial N}{\partial x} = \dfrac{(x^3+3xy^2)(y) - (xy)(6xy)}{(x^3+3xy^2)}$

General Solution:

$$\int \frac{x^2+2y^2}{x^3+3xy^2}\,dx + \int \frac{xy}{x^3+3xy^2}\,dy = c$$

$$\int \frac{x^2}{x^3+3xy^2}\,dx + \int \frac{2y^2}{x^2+3xy^2}\,dx + \int 0\,dx = c$$

$$\frac{x}{x^2+3y^2}\,dx - \frac{\ln\left(\frac{3y^3}{x^2}+1\right)+1}{3} = c$$

$$\therefore \frac{\ln(x^2+3y^2)}{2} - \frac{\ln\left(\frac{3y^3}{x^2}+1\right)+1}{3} = c$$

19.04.2023

Method 2: The differential equation $Mdx + Ndy = 0$ is of the form $y f(x,y)dx + x\, g(x,y)\, dy = 0$ and also non-homogenous.

Then Integrating factor is $I_F = \dfrac{1}{M\cancel{d}x - Ny}$

11Q Solve $y(1+xy)dx + x(1-xy)dy = 0$ — (1)

Eq (1) is in the form $Mdx + Ndy = 0$

where,

$M = y(1+xy)$ $\qquad\qquad$ $N = x(1-xy)$

$\dfrac{\partial M}{\partial y} = 1 + x(2y)$ $\qquad\qquad$ $\dfrac{\partial N}{\partial x} = 1 - y(2x)$

$= 1 + 2xy$ $\qquad\qquad$ $= 1 - 2xy$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq (1) is non Exact

Eq (1) is non homogenous.

Eq (1) is in the form $y f(x,y)dx + x\, g(x,y)dy = 0$.

Hence, According to method 2,

$$I_F = \frac{1}{M\cancel{d}x - N\cancel{d}y}$$

$$= \frac{1}{y(1+xy)x - x(1-xy)y}$$

$$I_F = \frac{1}{\cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2} = \frac{1}{2x^2y^2},$$

Multiplying IF and Eq ②

$$\frac{1}{2x^2y^2}\left[y(1+xy)dx + x(1-xy)dy\right] = 0$$

$$\frac{1+xy}{2x^2y}dx + \frac{1-xy}{2xy^2}dy = 0. \quad \text{— ②}$$

Eq ② is in form $Mdx + Ndy = 0$

where,

$$M = \frac{1+xy}{2x^2y} \qquad N = \frac{1-xy}{2xy^2}$$

$$\frac{\partial M}{\partial y} = \frac{2x^2y(x) - (1+xy)2x^2}{(2x^2y)^2} \qquad \frac{\partial N}{\partial x} = \frac{2xy^2(-y) - (1-xy)(2y^2)}{(2xy^2)^2}$$

$$= \frac{\cancel{2x^2}(\cancel{xy} - 1 - \cancel{xy})}{(\cancel{2x^2y})(2x^2y)} \qquad = \frac{\cancel{2y^2}(-\cancel{xy} - 1 + \cancel{xy})}{(\cancel{2xy^2})(2xy^2)}$$

$$= \frac{-1}{2x^2y^2} \qquad = \frac{-1}{2x^2y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq ② is exact

General Solution of Eq ②

$$\underset{y\text{ Const}}{\int M dx} + \underset{x\text{ eleminate}}{\int N dy} = C$$

$$\int \frac{1+xy}{2x^2y}dx + \int \frac{1-xy}{2xy^2}dy = c$$

$$\int \frac{1}{2x^2y}dx + \int \frac{xy}{2x^2y}dx + \int \frac{1}{2xy^2}dy - \int \frac{x^2y}{2xy^2}dy = c$$

$$\Rightarrow \frac{1}{2y} \int \frac{1}{x^2}dx + \frac{1}{2}\int \frac{1}{x}dx + 0dy - \frac{1}{2}\int \frac{1}{y}dy$$

$$\frac{-1}{2xy} + \frac{1}{2}\log(x) - \frac{1}{2}\log(y) = c$$

$\therefore \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$ is the general solution for the equation $y(1+xy)dx + x(1-xy)dy = 0$.

12Q Solve $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$ — (1)

Eq(1) is in form $Mdx + Ndy = 0$

where,
$M = (xy\sin xy + \cos xy)y$

$$\frac{\partial M}{\partial y} = \cancel{x(2y)(-\sin xy)x + (\sin xy)x}$$

$$= x\frac{d}{dy}(y^2\sin xy) + \frac{d}{dy}(y\cdot\cos xy)$$

$$= x\left[(y^2)(\cos xy)(x) + (2y)(\sin xy)\right] + \left[y^2(-\sin xy)(x) + (2y)\cos xy\right]$$

$$= x\left[x\,y^2\cos xy + 2y\sin xy - xy^2\sin xy + 2y\cos xy\right]$$

$$= x\left[y\sin xy\,(2 - xy) + y\cos xy\,(xy + 2)\right]$$

$$\frac{\partial M}{\partial y} = x^2y^2\cos xy + 2xy\sin xy - xy\sin xy + \cos xy$$

$$N = (xy\sin xy \mp \cos xy)\,x$$

$$\frac{\partial N}{\partial x} = x^2y\sin xy \mp x\cos xy$$

$$= y\,\frac{d}{dx}(x^2\sin xy) \mp \frac{\partial}{\partial x}(x\cos xy)$$

$$= y\left[x^2(y\cos xy) + (2x)\sin xy\right] \mp x(y\sin xy)(-1) \mp (1)\cos xy$$

$$= x^2y^2\cos xy + 2xy\sin xy + xy\sin xy \mp \cos xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq ① is not Exact

Eq ① is non homogenous.

Eq ① is in form $y\,f(x,y)\,dx + x\,f(x,y)\,dy = 0$

$$I_F = \frac{1}{Mx - Ny}$$

$$= \frac{1}{y(xy\sin xy + \cos xy)x - y(xy\sin y - \cos xy)x}$$

$$= \frac{1}{xy(\cancel{xy\sin xy} + \cos xy - \cancel{xy\sin xy} + \cos xy)}$$

$$= \frac{1}{2xy\cos xy}$$

Multiplying I_F and Eq ①

$$\frac{1}{2xy\cos xy}\left[(xy\sin xy + \cos xy)y\,dx + (xy\sin y - \cos xy) \times dy\right] = 0$$

$$\left(\frac{y.\tan xy}{2} + \frac{1}{2x\cancel{y}}\right)dx + \left(\frac{x.\tan xy}{2} - \frac{1}{2\cancel{x}y}\right)dy = 0 \quad \text{--- ②}$$

Eq ② is in form $Mdx + Ndy$.

$$M_1 = y.\frac{\tan xy}{2} + \frac{1}{2x\cancel{y}}$$

$$\frac{\partial M}{\partial y} = \cancel{\frac{1}{2}(\sec^2 xy)x} + \cancel{\frac{1}{2x}\left(\frac{-1}{xy^2}\right)} = \frac{xy(\sec^2 xy) + \tan xy}{2}$$

$$= \cancel{x.\frac{\sec^2 xy}{2}} + \cancel{\frac{-1}{2xy^2}} = \cancel{\frac{x^2y^2\sec^2 xy - 1}{2xy^2}}$$

$$\frac{\partial M}{\partial y} = \frac{xy\sec^2 xy + \tan xy}{2}$$

$$N_1 = x.\frac{\tan xy}{2} - \frac{1}{2\cancel{x}y^2}$$

$$\frac{\partial N}{\partial x} = \cancel{\frac{y\sec^2 xy}{2}} \cancel{-\frac{1}{2y^2}\left(\frac{-1}{x^2}\right)} = \frac{xy\sec^2 xy + \tan xy}{2}$$

$$\frac{\partial N}{\partial x} = \frac{xy\sec^2 xy + \tan xy}{2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}.$$

Eq ② is Exact.

General Solution of Eq ② is.

$$\int_{y\ \text{const}} \frac{y\ \tan xy}{2} + \frac{1}{2x}\, dx + \int_{x\ \text{elemt.}} \frac{x\ \tan xy}{2} + \frac{1}{2y}\, dy = c$$

$$= \frac{y}{2}\int \tan xy\, dx + \frac{1}{2}\int \frac{1}{x} dx + \frac{1}{2}\int \frac{1}{y} dy = c$$

$$= -\frac{1}{2}\log|\cos xy| + \frac{1}{2}\log|x| - \frac{1}{2}\log|y| = c$$

$\therefore\ -\frac{1}{2}\log|\cos xy| + \frac{1}{2}\log(x) - \frac{1}{2}\log(y) = c$ is the

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20.04.2023

139 Solve $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$ — (1)

Eq (1) is in form $M\,dx + N\,dy = 0$

such that,

$M = y - xy^2$ $\qquad$ $N = -x - x^2y$

$\frac{\partial M}{\partial y} = 1 - x(2y)$ $\qquad$ $\frac{\partial N}{\partial x} = -1 - y(2x)$

$= 1 - 2xy.$ $\qquad$ $= -1 - 2xy.$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq (1) is non exact

Eq (1) is non homogenous.

Eq (1) is in form $y f(x,y)\,dx + x g(x,y)\,dy = 0$

$$I_F = \frac{1}{Mx - Ny}$$

$$= \frac{1}{(y - xy^2)x - (x^2 + x^2y)y}$$

$$= \frac{1}{xy - x^2y^2 - xy - x^2y^2}$$

$$I_F = \frac{1}{2xy}$$

Multiplying I_F and Eq ①

$$\frac{y(1-xy)}{2xy}\,dx - \frac{x(1+xy)}{2xy}\,dy = 0$$

$$\frac{1-xy}{2x}\,dx - \frac{1+xy}{2y}\,dy = 0 \quad — ②$$

Eq ② is in form $M\,dx + N\,dy = 0$

$M_1 = \frac{1}{2x} - \frac{xy}{2x}$ $\qquad N_1 = -\left(\frac{1}{2y} + \frac{xy}{2y}\right)$

$$\frac{\partial M_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2x} - \frac{y}{2}\right) \qquad \frac{\partial N_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2y} + \frac{x}{2}\right)(-1)$$

$$= \cancel{\frac{1}{2}\left(\frac{-1}{x^2}\right)} - \frac{1}{2} \qquad = \left(\frac{1}{2y}(0) + \frac{1}{2}(1)\right)(-1)$$

$$= \frac{1}{2x}(0) - \frac{1}{2}(1) \qquad = \frac{-1}{2}$$

$$= \frac{-1}{2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq ② is exact

General Solution of Eq (2) is

$$\underset{y\ \text{const}}{\int M_1 dx} + \underset{x\ \text{eliminate}}{\int N_1 dy} = c$$

eliminated

$$\int\left(\frac{1}{2x} - \frac{y}{2}\right)dx - \left(\int\left(\frac{1}{2y} + \frac{x}{2}\right)dy\right) = c$$

$$\frac{1}{2}\int\frac{1}{x}dx - \frac{y}{2}\int 1\,dx - \frac{1}{2}\int y\,dy = c$$

$$\frac{1}{2}\log(x) - \frac{y}{2}(x) - \frac{1}{2}\left(\frac{y^2}{2}\right) = c$$

$\therefore\ \log(x) - xy - \frac{y^2}{2} = 2c$ is the general solution for $(y - xy^2)\,dx - (x + x^2y)\,dy = 0$.

Method 3: The differential equation of the form $M dx + N dy = 0$ is non-homogenous and equation is not in the form of $y f(x,y)\,dx + x g(x,y)\,dy$

Then, I_F (Integrating Factor is) $= e^{\int f(x)}$

where $f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$

Method 4: The differential Equation $M dx + N dy = 0$ is ~~and is hom~~ non homogenous and Equation is not in the form of $y f(x,y)\,dx + x g(x,y)\,dy = 0$ then,

$I_F = e^{\int g(y)}$

where $g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$

Eq ① is in form $Mdx + Ndy = 0$

where,

$M = x^2 + y^3 + 6x$

$\frac{\partial M}{\partial y} = x^2(0) + 3y^2 + 6x(0)$

$= 3y^2$

$N = y^2 x$

$\frac{\partial N}{\partial x} = \cancel{2} y^2(\cancel{1})$

$= \cancel{2xy}\, y^2$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq ① is non-exact

Eq ① is non-homogenous

Eq ① is not in form $y f(x,y)dx + x\, g(x,y)dy = 0$

Since N is small,

$\Rightarrow f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$

$= \frac{1}{y^2 x}\left[3y^2 - \cancel{2xy}\, y^2\right]$

$= \cancel{\frac{y^2(3-1)}{y^2 x}} = \frac{2y^2}{y^2 x}$

$= \frac{2}{x}$

$I_F = e^{\int \frac{2}{x} dx}$

$e^{2\log x}$

$I_F = e^{\log x^2}$

$I_F = x^2$

Multiplying I.F with Eq (1)

$$x^2\left[x^2+y^3+6x\right]dx + x^2\left[y^2x\right]dy = 0$$

$$(x^4+x^2y^3+6x^3)dx + x^3y^2\,dy = 0 \quad \text{--- (2)}$$

Eq (2) is in form $M\,dx + N\,dy = 0$

where

$M = x^4+x^2y^3+6x^3$

$\frac{\partial M}{\partial y} = x^4(0) + x^2(3y^2) + 6x^3(0)$

$= 3x^2y^2$

$N = x^3y^2$

$\frac{\partial N}{\partial x} = y^2(3x^2)$

$= 3x^2y^2$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq (2) is exact.

General Solution of Eq (2) is

$$\int_{y\ \text{const}} M\,dx + \int_{x\ \text{eleminate}} N\,dy = c$$

$$\int (x^4+x^2y^3+6x^3)\,dx + \int_{x\ \text{eleminate}} x^3y^2\,dy = c$$

$$\frac{x^5}{5} + y^3\left(\frac{x^3}{3}\right) + \frac{6x^4}{4} + 0\,dy = c$$

$\therefore \frac{x^5}{5} + \frac{x^3y^3}{3} + \frac{3x^4}{2} = c$ is the general solution of the equation $(x^2+y^3+6x)\,dx + y^2x\,dy = 0$.

21.04.2023

159 Solve $y(xy+e^x)dx - e^x dy = 0$ — (1)

Solution -

Eq (1) is in form $Mdx + Ndy = 0$

where,

$M = y(xy+e^x)$.

$\frac{\partial M}{\partial y} = x(2y) + (e^x)(1)$

$= 2xy + e^x$

$N = -e^x$

$\frac{\partial N}{\partial x} = -1(e^x)$

$= -e^x$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq (1) is non exact

Eq (1) is non-homogenous

Eq (1) is not in form $y f(x,y)dx + x g(x,y)dy = 0$.

Since, N is small,

$$f(x) = \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$$

$$= \frac{1}{-e^x}\left[2xy + e^x - (-e^x)\right]$$

$$= \frac{-2xy + e^x - e^x}{e^x}$$

Not possible with $f(x)$.

$$g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$= \frac{1}{y(xy + e^x)}\left[-e^x - \left[2xy + e^x\right]\right]$$

$$= \frac{-2\cancel{(xy + e^x)}}{y\cancel{(xy + e^x)}}$$

$$= \frac{-2}{y}$$

$$\Rightarrow I_F = e^{\int \frac{-2}{y} dy}$$

$$= e^{-2\int \frac{1}{y} dy}$$

$$= e^{-2(\log y)}$$

$$= e^{\log y^{-2}}$$

$$I_F = y^{-2}$$

Multiplying I_F and Eq ①

$$(y^{-2})y(xy + e^x)\,dx - y^{-2}e^x dy = 0$$

$$\left(\frac{xy + e^x}{y}\right)dx - \frac{e^x}{y^2}dy = 0 \quad — ②$$

Eq ② is in form $M dx + N dy = 0$

where,

$M = \left(\frac{xy}{y} + \frac{e^x}{y}\right)$ | $N = -\frac{e^x}{y^2}$

$$\frac{\partial M}{\partial y} = x(0) + e^x\left(\frac{-1}{y^2}\right) \qquad \frac{\partial N}{\partial x} = \frac{-1}{y^2}(e^x)$$

$$= \frac{e^x}{y^2} \qquad = \frac{-e^x}{y^2}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq ② is exact

General solution of Eq ② is

$$\underset{y\text{ constant}}{\int M\,dx} + \underset{x\text{ eleminate}}{\int N\,dy} = c$$

$$\underset{y\text{ const}}{\int\left(x + \frac{e^x}{y}\right)dx} + \underset{x\text{ eleminati}}{\int \frac{-e^x}{y^2}\,dy} = c$$

$$\int x\,dx + \frac{1}{y}\int e^x\,dx + \int 0\,dy = c$$

$\therefore \frac{x^2}{2} + \frac{e^x}{y} = c$ is the ~~geral~~ general solution of the equation $y(xy + e^x)dx - e^x dy = 0$.

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169 Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$ — ①

solution :-

Eq ① is in form $M dx + N dy = 0$

where

$$M = y^4 + 2y \qquad N = xy^3 + 2y^4 - 4x$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \qquad \frac{\partial N}{\partial x} = (1)y^3 + 2y^4(0) - 4$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq ① is non exact

Eq ① is non homogenous and not in form

$y f(x,y)dx + x g(x,y)dy = 0.$

Since M is small,

$$g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$= \frac{1}{y^4+2y}\left[(y^3-4) - (4y^3+2)\right]$$

$$= \frac{-3y^3-6}{y(y^3+2)} = \frac{-3(y^3+2)}{y(y^3+2)}$$

$$g(y) = \frac{-3}{y}$$

$$I_F = e^{\int g(y)}$$

$$= e^{\int \frac{-3}{y} dy}$$

$$= e^{-3 \log y}$$

$$= y^{-3}.$$

Multiplying ~~b~~ I_F and Eq ①

$$\frac{1}{y^3}(y^4+2y)dx + \frac{xy^3+2y^4-4x}{y^3}dy = 0$$

$$\left(y + \frac{2}{y^2}\right) dx + \left(x + 2y - \frac{4x}{y^3}\right) dy = 0 \quad - (2)$$

Eq (2) is in form $M dx + N dy = 0$

$M = y + \frac{2}{y^2}$

$\frac{\partial M}{dy} = 1 + 2 \frac{(-2)}{y^3}$

$= 1 + \left(-\frac{4}{y^3}\right)$

$N = x + 2y - \frac{4x}{y^3}$

$\frac{\partial N}{\partial x} = 1 + 2y(0) - \frac{4}{y^3}(1)$

$= 1 + \left(\frac{-4}{y^3}\right)$

$$\frac{\partial M}{dy} = \frac{\partial N}{\partial x}$$

Eq (2) is exact

General solution of Eq (2) is

$$\underset{y \text{ constant}}{\int M dx} + \underset{x \text{ eleminate}}{\int N dy} = c$$

$$\Rightarrow \underset{y \text{ const}}{\int \left(y + \frac{2}{y^2}\right) dx} + \underset{x \text{ eleminate}}{\int \left(x + 2y - \frac{4x}{y^3}\right) dy} = c$$

$$\Rightarrow y \int 1 dx + \frac{2}{y^2} \int 1 dx + \int 2y \, dy = c$$

$$\Rightarrow xy + \frac{2x}{y^2} + \cancel{2} \frac{y^2}{\cancel{2}} = c$$

$\therefore xy + \frac{2x}{y^2} + y^2 = c$ is the general solution of Equation

17A Solution :-

$M = 2y$ | $N = 2x\log x - xy$

$\frac{\partial M}{\partial y} = 2(1)$

$= 2$

$\frac{\partial N}{\partial x} = 2x\left(\frac{1}{x}\right) + \log x(2) - y(1)$

$= 2 + 2\log x - y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eq(1) is non exact, non homogenous

not in form $y f(x,y)dy + x g(x,y)dx$.

M is small,

$$g(y) = \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right]$$

$$= \frac{1}{2y}\left[\cancel{2} + 2\log x - y - \cancel{2}\right]$$

$$= \frac{2\log x - y}{2y}$$

$$f(x) = \frac{1}{2x\log x - xy}\left[2 - (2 + 2\log x - y)\right]$$

$$= \frac{-\cancel{1}(2\log x - y)}{x(2\log x - y)}$$

$$f(x) = -\frac{1}{x}$$

$$I_f = e^{\int f(x)}$$

$$e^{-1\int \frac{1}{x}dx}$$

$$e^{\log x^{-1}} = x^{-1}.$$

Multiplying I_f and Eq ①

$$\frac{2x\log x - xy}{x}\, dy + \frac{2y}{x}\, dx = 0$$

Eq ② is in form $M\,dx + N\,dy = 0$

where,

$N = 2\log x - y$

$M = \frac{2y}{x}$

$$\frac{\partial N}{\partial x} = 2\left(\frac{1}{x}\right) - y(0)$$

$$= \frac{2}{x}$$

$$\frac{\partial M}{\partial y} = \frac{2}{x}(1)$$

$$= \frac{2}{x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Eq ② is exact

General Solution of Eq ② is

$$\int\limits_{y\text{ constant}} M\,dx + \int\limits_{x\text{ eleminat}} N\,dy = c$$

$$\int\limits_{y\text{ const}} \frac{2y}{x}\, dx + \int\limits_{x\text{ elem.}} (2\log x - y)\, dy = c$$

$$2y\int \frac{1}{x}\, dx + (-1)\int y\, dy = c$$

$\therefore\ 2y(\log x) - \frac{y^2}{2} = c$ is the general

solution of Equation $(2x\log x - xy)\,dy + 2y\,dx = 0$.

24.04.2023

3

Linear Differential Equation of First Order

An equation of the form $\frac{dy}{dx} + y P(x) = Q(x)$ where P and Q are either constants or function of x only is called linear Differential Equation of first order in x.

(or)

$$\frac{dx}{dy} + x P(y) = Q(y)$$

- Working Rule: An equation can be written

$$\frac{dy}{dx} + y P(x) = Q(x)$$

- Find Integrating Factor $(I_F) = e^{\int P(x) dx}$
- write the general solution

$$y \cdot I_F = \int I_F Q(x) dx + c$$

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189 Find the Integrating Factor

$$(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2 \quad \text{— (1)}$$

Solution:

$$\left((x+1)\frac{dy}{dx} - y\right)\frac{1}{(x+1)} = e^{3x}(x+1)^2\left(\frac{1}{x+1}\right)$$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1) \quad \text{— (2)}$$

Eq (2) is in form $\frac{dy}{dx} + y P(x) = Q(x)$

Where,

$$P(x) = \frac{-1}{x+1}, \qquad Q(x) = e^{3x}(x+1)$$

$$I_F = e^{\int P(x)\,dx}$$

$$= e^{\int \left(\frac{-1}{x+1}\right) dx}$$

$$= \cancel{e^{(-1)\int \frac{1}{x}dx - \int +1\,dx}}$$

$$= \cancel{e^{-1(\log x) - x}} \quad e^{-1\left(\int \frac{1}{x+1}dx\right)}$$

$$= e^{-1(\log|x+1|)}$$

$$= e^{\log(x+1)^{-1}}$$

$$= (x+1)^{-1}$$

$\therefore I_F = \frac{1}{x+1}$ is the Integrating factor for the equation $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

General solution: $y\, I_F = \int I_F \cdot Q(\cancel{y}) \cdot dx + c$

$$y\left(\frac{1}{x+1}\right) = \int \left(\frac{1}{x+1}\right) e^{3x}(x+1)\,dx + c$$

$$\frac{y}{x+1} = \int e^{3x}\,dx + c$$

$$\frac{y}{x+1} = \frac{e^{3x}}{3} + c$$

794 Solve $(x+2y^3)\frac{dy}{dx} = y$

$$\frac{dy}{dx} = \frac{y}{x+2y^3}$$

$$\frac{dx}{dy} = \frac{x+2y^3}{y}$$

$$\frac{dx}{dy} + x\left(\frac{-1}{y}\right) = \frac{2y^3}{y} = 2y^2 \quad — (1)$$

Eq (1) is in form $\frac{dx}{dy} + x\,P(y) = Q(y)$

where,

$$P(y) = \frac{-1}{y} \qquad Q(y) = 2y^2.$$

Hence, $I_F = e^{\int P(y)\,dy}$

$$= e^{-1\int \frac{1}{xy} dy}$$

$$= e^{-1\log(xy)}$$

$$= e^{\log(xy)^{-1}}$$

$$\therefore I_F = \frac{1}{xy}$$

General Solution is : $x\,I_F = \int I_F\,Q(y)\,dy + c$

$$x\left(\frac{1}{xy}\right) = \int \frac{1}{xy}(2y^2)\,dy + c$$

$$\cancel{1 = \frac{x}{xy}\left(\frac{2y^3}{3}\right) + c} \qquad \frac{x}{y} = 2\int y\,dy + c$$

~~$\therefore 1 = \frac{2y^3}{3x} + c$ is the general solution~~

$\therefore \frac{x}{y} = y^2 + c$ is the general solution of equation

$$(x+2y^3)\frac{dy}{dx} = y.$$

20Q Solve $x\frac{dy}{dx} + y = \log x$

Solution :-

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{x}\right) = \frac{\log x}{x} \quad \text{—①}$$

Eq ① is in form $\frac{dy}{dx} + y\,P(x) = Q(x)$

where,

$$P(x) = \frac{1}{x} \qquad Q(x) = \frac{\log x}{x}$$

$$I_F = e^{\int P(x)dx}$$

$$= e^{\int \frac{1}{x}dx}$$

$$= e^{\log(x)}$$

$$I_F = x$$

General Solution : $y\,I_F = \int I_F\,Q(x) + c$

$$y(x) = \int x\left(\frac{\log x}{x}\right)dx + c$$

$$xy = \int \log x\,dx + c$$

$\therefore \quad xy = x(\log x - 1) + c$ is the general solution of the equation $x\frac{dy}{dx} + y = \log x$.

21Q Solve $(1-x^2)\frac{dy}{dx} + xy = ax$

$$\frac{dy}{dx} + \frac{xy}{(1-x^2)} = \frac{ax}{(1-x^2)}$$

$$\frac{dy}{dx} + y\left(\frac{x}{(1-x^2)}\right) = \frac{ax}{1-x^2} \quad \text{—①}$$

$$P(x) = \frac{x}{1-x^2} \qquad Q(x) = \frac{ax}{(1-x^2)}$$

$$I_F = e^{\int P(x)\,dx} \qquad \because \int \frac{f'(x)}{f(x)} = \log|f(x)|$$

$$= e^{-\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx}$$

$$= e^{-\frac{1}{2}\log|1-x^2|}$$

$$= e^{\log(1-x^2)^{-\frac{1}{2}}}$$

$$= (1-x^2)^{-\frac{1}{2}}.$$

$$= \frac{1}{\sqrt{1-x^2}}$$

General Solution : $y\, I_F = \int I_F\, Q(x)\, dx + c$

$$y \frac{1}{\sqrt{1-x^2}} = \int \frac{1}{\sqrt{1-x^2}} \frac{ax}{(1-x^2)}\, dx + c$$

$$\frac{y}{\sqrt{1-x^2}} = \int \frac{ax}{(1-x^2)^{3/2}}\, dx + c$$

$$\left[\because \int [f(x)]^n f'(x) = \frac{[f(x)]^{n+1}}{n+1}\right] = a\int (1-x^2)^{-3/2}\, x\, dx + c$$

$$= \frac{-a}{2}\int (1-x^2)^{-3/2}(-2x)\, dx + c$$

$$= \left(\frac{-a}{2}\right)\frac{(1-x^2)^{-3/2+1}}{\frac{-3}{2}+1}$$

$\therefore \dfrac{y}{\sqrt{1-x^2}} = \dfrac{-a}{\sqrt{1-x^2}} + c$ is the general Solution.

Solve $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0 \rightarrow ①$

$$\frac{dr}{d\theta} = r(2\cot\theta) = -\sin 2\theta \quad -①$$

Eq ① is in form $\frac{dr}{d\theta} + r\,P(\theta) = Q(\theta)$

where,

$P(\theta) = 2\cot\theta \qquad Q\theta = -\sin 2\theta$

$$I_F = e^{\int P(\theta)d\theta}$$

$$= e^{\int 2\cot\theta\, d\theta}$$

$$= e^{2(\log\sin\theta)}$$

$$= e^{\log\sin^2\theta}$$

$I_F = \sin^2\theta$

General solution : $r \cdot I_F = \int I_F\, Q(\theta)d\theta + c$

$$r\sin^2\theta = \int \sin^2\theta(-\sin 2\theta)\,d\theta + c$$

$$= \int \sin^2\theta(-2\sin\theta\cos\theta)\,d\theta + c$$

$$= -2\int \sin^3\theta \cdot \cos\theta + c$$

Let $\sin\theta = t$

$\Rightarrow \cos\theta\, d\theta = dt$

$= -2\int t^3 dt + c$

$= -2\frac{t^4}{4} + c$

$\therefore r\sin^2\theta + \frac{2\sin^4\theta}{4} + c$

is the general solution.

Solve $x \log x \frac{dy}{dx} + y = 2 \log x$

$$\frac{dy}{dx} + y\left(\frac{1}{x \log x}\right) = \frac{2 \log x}{x \log x}$$

$$\frac{dy}{dx} + y\left(\frac{1}{x \log x}\right) = \frac{2}{x} \quad \text{— (1)}$$

Eq (1) is in form $\frac{dy}{dx} + y P(x) = Q(x)$

where, $P(x) = \frac{1}{x \log x}$ $\quad Q(x) = \frac{2}{x}$

$$I_F = e^{\int P(x) dx}$$

$$= e^{\int \frac{1}{x \log x} dx}$$

$$= e^{\int \frac{1}{x} \frac{1}{\log x} dx} \quad \left[\log x = t, \frac{1}{x} dx = dt\right]$$

$$= e^{\int \frac{1}{t} dt}$$

$$I_F = e^{\log t} = t = \log x$$

General Solution: $y I_f = \int I_f Q(x) dx + c$

$$y(\log x) = \int (\log x) \frac{2}{x} dx + c$$

$$y \log x = 2 \int t \, dt + c$$

$$y \log x = 2 \frac{t^2}{2} + c$$

$\therefore y \log x = (\log x)^2 + c$ is the general solution.

$$\frac{dy}{dx} + (y-1)\cos x = e^{-\sin x}\cos^2 x.$$

$$\frac{dy}{dx} + y(\cos x) = e^{-\sin x}\cos^2 x - \cos x \quad \text{—①}$$

Eq ① is in form $\frac{dy}{dx} + y\,P(x) = Q(x)$

where, $P(x) = \cos x$ $\quad Q(x) = e^{-\sin x}\cos^2 x - \cos x.$

$$I_F = \cancel{\cos x}\; e^{\int P(x)dx}$$

$$= e^{\int \cos x\, dx}$$

26.04.2023

Bernoullis Equation:

An equation of the form $\frac{dy}{dx} + y\,P(x) = y^n\,Q(x)$

where, P, Q are constants

$n+1$

. **Working Rule:**

- Convert Bernoullis equation to linear by following the given procedure.

$$\frac{dy}{dx} + y \cdot P(x) = y^n Q(x)$$

divide by y^n.

$$\frac{1}{y^n}\frac{dy}{dx} = y^{1-n} P(x) = Q(x)$$

Let $y^{1-n} = t$ =>

$$(1-n)\, y^{1-n-1}\frac{dy}{dx} = \frac{dt}{dx}$$

$$(1-n)\, y^{n}\frac{dy}{dx} = \frac{dt}{dx}$$

=> $\cancel{\frac{1}{y^n}}\cancel{(1-n)}\cancel{y^n}\cancel{\frac{dy}{dx}}$ $y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$

$$\frac{1}{1-n}\cdot\frac{dt}{dx} + t\,P(x) = Q(x)$$

$$\frac{dt}{dx} + (1-n)\,t\,P(x) = Q(x)(1-n)$$

$$\frac{dt}{dx} = t P_1(x) = Q_1(x) \quad — ②$$

· Find Integrating factor for Eq②

$$\cdot I_F = e^{\int P_1(x)}$$

· General solution for Eq②

$$t I_F = \int I_F \cdot Q_1(x) dx + C \quad — ③$$

Substitute t (value) in Eq③ to get G.S for Eq①.

$$\frac{dt}{dx} + \frac{(-n)\, t^{1/n}}{t^{\frac{1}{n}-1}} P(x) = (-n)^{1/n} Q(x)$$

$$\frac{dt}{dx} = (-n)\, t P(x) = -n\, t^{1/n} Q(x)$$

Q Solve $\frac{dy}{dx} + xy = xy^2 \quad — ①$

$$\frac{1}{y^2}\frac{dy}{dx} + x y^{1-2} = x$$

$$\frac{1}{y^2}\frac{dy}{dx} + y^{-1} x = x \quad — ②$$

Let $y^{-1} = t$

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dt}{dx}(-y^2)$$

② $\Rightarrow \frac{1}{y^2}(-y^2)\frac{dt}{dx} + tx = x$

$$-\frac{dt}{dx} + tx = x$$

$$\frac{dt}{dx} + t(-x) = -x \quad \text{— ③}$$

Eq ③ is in form $\frac{dt}{dx} + t(P(x)) = Q(x)$

$\Rightarrow P(x) = -x \qquad Q(x) = -x$

$$I_F = e^{\int P(x)\,dx}$$

$$= e^{\int -x\,dx}$$

$$= e^{-1\left(\frac{x^2}{2}\right)}$$

$$= e^{\frac{-x^2}{2}}$$

GS of Eq ③ : $t\,I_F = \int I_F\,Q(x)\,dx + c$

$$t\left(e^{-x^2/2}\right) = \int e^{-x^2/2}(-x)\,dx$$

$t\left(e^{-x^2/2}\right) = \int e^z\,dz$

$t\,e^{-x^2/2} = e^z$

$y^{-1}e^{\frac{-x^2}{2}} = e^{\frac{-x^2}{2}}dx + c$

is the general solution

$$\cancel{= -1\left[e^{-x^2/2}\left(\frac{x^2}{2}\right) + (-x)e^{-x^2/2}\left(\frac{-x^3}{6}\right)\right]}$$

$$= e^{-x^2/2}\left[\frac{-x^2}{2} + \frac{(-x)(-x^3)}{6}\right]$$

9 Solve $x \frac{dy}{dx} + y = x^3 y^6$

$$\frac{dy}{dx} + y x^{-1} = x^2 y^6$$

$$\frac{1}{y^6} \frac{dy}{dx} + y^{-5} x^{-1} = x^2$$

Let $y^{-5} = t$. $\Rightarrow \frac{1}{y} = t^{\frac{1}{5}}$

$$\frac{-5}{y^6} \cdot \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{dt}{dx}\left(\frac{y^6}{-5}\right)$$

$$\cancel{t^{\frac{6}{5}}}\ \cancel{\frac{1}{y^6}} \frac{dt}{dx} \frac{\cancel{y^6}}{-5} + t x^{-1} = x^2$$

$$\frac{dt}{dx} + (-5x^{-1}) t = -5x^2 \quad — ①$$

Eq ① is in form $\frac{dt}{dx} + t\,P(x) = Q(x)$

where, $P(x) = \frac{-5}{x}$, $Q(x) = -5x^2$

$$I_F = e^{\int \frac{-5}{x} dx}$$

$$= e^{-5 \int \frac{1}{x} dx}$$

$$e^{-5 \log|x|}$$

$$= e^{\log x^{-5}}$$

$$= x^{-5}$$

$$= \frac{1}{x^5}$$

General Solution is $t \cdot I.F = \int I.F \; Q(x)\, dx$

$$\frac{t}{x^5} = \int \frac{1}{x^5}(-5x^2)\, dx$$

$$\frac{t}{x^5} = -5\int \frac{1}{x^3}\, dx$$

$$\Rightarrow \frac{t}{x^5} = -5 \cancel{\log}\left(\frac{-1}{2x^2}\right) + c$$

$\therefore \frac{t}{x^5} = \frac{5}{2x^2} + c$ is the general solution of

the equation $x\frac{dy}{dx} + y = x^3 y^6$.

28.04.2023

Its in Bernoullis Equation

Q Solve $\frac{dy}{dx} + y\tan x = y^2 \sec x$ $\left[\frac{dy}{dx} + yP(x) = y^n(Q(x))\right]$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \quad \text{—} (1)$$

Let $\boxed{\frac{1}{y} = t} \Rightarrow y = \frac{1}{t}$

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\boxed{\frac{dy}{dx} = -y^2\frac{dt}{dx}}$$

$$(1) \Rightarrow \frac{1}{\cancel{y^2}}(-\cancel{y^2})\frac{dt}{dx} + \cancel{\frac{1}{t}}\; t(\tan x) = \sec x$$

$$\frac{dt}{dx} + t(-\tan x) = -\sec x \quad \text{—} (2)$$

Eq ② is in form $\frac{dt}{dx} + t\,P(x) = Q(x)$

where, $P(x) = -\tan x$ $\quad Q(x) = -\sec x$

$$I_f = e^{\int -\tan x\,dx} \quad \left[\int \tan x\,dx = \log \sec x + c\right]$$

$$= e^{-1\int \tan x\,dx}$$

$$I_f = e^{-1(\log \sec x)}$$

$$= \sec^{-1} x$$

$$I_f = \cos x$$

General Solution: $t\,I_f = \int I_f\,Q(x)\,dx + c$

$$t(\cos x) = \int \cos x(-\sec x)\,dx + c$$

$$t\cos x = -1\int \frac{\sec x}{\sec x}\,dx + c$$

$$t\cos x = -1\left(\int 1\,dx\right) + c$$

$\therefore \quad t\cos x = -x + c$ is the general solution for eq ②

$\therefore \quad \frac{\cos x}{y} = -x + c$ is the general solution for equation $\frac{dy}{dx} + y\tan x = y^2\sec x$.

Q Solve $\frac{dy}{dx} + yx + y^2 e^{\frac{x^2}{2}}\sin x = 0$

$$\frac{dy}{dx} + y(x) = y^2\left(-e^{\frac{x^2}{2}}\sin x\right)$$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}(x) = -e^{\frac{x^2}{2}}\sin x$ — ①

Let $\frac{1}{y} = t$

$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dy}{dx} = \frac{dt}{dx}(-y^2)$

① => $\cancel{\frac{1}{y^2}}\frac{dt}{dx}(-\cancel{y^2}) + t(x) = -e^{\frac{x^2}{2}}\sin x$

$\frac{dt}{dx} + t(-x) = e^{x^2/2}\sin x$ — ②

Eq ② is in form $\frac{dy}{dx} + y(P(x)) = Q(x)$

where,

$P(x) = -x$ $\qquad Q(x) = e^{\frac{x^2}{2}}\sin x$

$I_F = e^{\int P(x)\,dx}$

$= e^{\int -x\,dx}$

$= e^{-1\left(\frac{x^2}{2}\right)}$

$= e^{\frac{-x^2}{2}}$

$t\,I_F = \int I_F\, Q(x)\,dx + c$

General Solution ~~$= \frac{dt}{dx} + t(P(x)) = Q(x)$~~

$t\,e^{-\frac{x^2}{2}} = \int e^{-x^2/2}\left(e^{\frac{x^2}{2}}\right)\sin x\,dx + c$

$t\,e^{\frac{-x^2}{2}} = \int e^{\frac{x^2-x^2}{2}}\sin x\,dx + c$

$$t\, e^{\frac{-x^2}{2}} \qquad \int \sin x \, dx + c$$

$$\therefore \frac{e^{\frac{-x^2}{2}}}{y} = -\cos x + c$$ is the general

solution of $\frac{dy}{dx} + yx + y^2 e^{\frac{x^2}{2}} dx = 0$.

Q Solve : $\frac{dy}{dx} - y\cos x = y^4(\sin^2 x - \cos x)$

Eq (1) is in form in $\frac{dy}{dx} + y P(x) = y^4 Q(x)$

$$\frac{3}{y^4}\frac{dy}{dx} - \frac{1}{y^3}\cos x = \sin^2 x - \cos x$$

Let $\boxed{\frac{1}{y^3} = t}$

$$-\frac{3}{y^4}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{dt}{dx}\left(\frac{-y^4}{3}\right)}$$

$$\frac{\cancel{3}}{\cancel{y^4}}\frac{dt}{dx}\left(\frac{-\cancel{y^4}}{\cancel{3}}\right) - t\cos x = \sin^2 x - \cos x$$

$$\cancel{\frac{dy}{}} \frac{dt}{dx} + t(\cos x) = \cos x - \sin^2 x \quad (2)$$

Eq (2) is in form $\frac{dy}{dx} + y P(x) = Q(x)$

where.

$P(x) = \cos x \qquad\qquad Q(x) = \cos x - \sin^2 x$

$$I_F = e^{\int P(x) dx} = e^{\int \cos x \, dx} = e^{\sin x}$$

$I_F = e^{\sin x}$

General Solution: $t\, I_F = \int I_F\, \phi(x)\, dx$

$$t \cdot e^{\sin x} = \int e^{\sin x} (\cos x - \sin 2x)\, dx$$

$$= \int e^{\sin x} \cos x\, dx - 2\int e^{\sin x} \sin x \cos x\, dx \quad \text{--- (3)}$$

$\boxed{\sin x = t}$

$\cos x\, dx = dt$

$\boxed{dx = \frac{dt}{\cos x}}$

$\int f g' = fg - \int f' g$

$f = t,\ g = e^t$

$f' = 1$

(3) =>

$$t\, e^{\sin x} = \int e^{t} \cancel{\cos x} \frac{dt}{\cancel{\cos x}} - 2\int e^{t} (\sin x) \cancel{\cos x} \frac{dt}{\cancel{\cos x}}$$

$$= \cancel{\int e^t dt - 2\int e^t \sin x\, t\, dt}$$

$$= \int e^{h} dh - 2\int e^{t} \cdot t\, dt$$

$$= e^{t} - 2\left[(t \cdot e^{t}) - \int 1 (e^{t})\, dt\right]$$

$$e^{t} - 2(t \cdot e^{t} - e^{t}) + C$$

$$t (e^{\sin x}) = e^{t}(1 - 2t + 2) + C$$

$$\frac{e^{\sin x}}{y^3} = e^{\sin x}(1 - 2\sin x + 2) + C$$

$(x-1)e^x$

$\int uv = uv_1 - u'v_2 - u''v_3 \dots$

$u = x \quad v_1 = e^x$

$u' = 1 \quad v_2 = e^x$

$u'' = 0 \quad v_3\, e^x$

$$\therefore \frac{e^{\sin x}}{y^3} = e^{\sin x}(3 - 2\sin x) + C$$

$\therefore \frac{1}{y^3} = 3 - 2\sin x$ is the general solution.

Q Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^{x} \sec y$

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\sin y}{\cos x\,(1+x)}\left(\frac{1}{\sec y}\right) = (1+x)e^{x}$$

$\frac{\sin y}{1+x} = e^{x} + c$

$\left[\frac{1}{\sec x} = \cos x\right]$

~~Cos~~

$$\cos y \frac{dy}{dt} - \frac{\sin y}{(1+x)} = (1+x)e^{x}$$

Let $\sin y = t$

$$\cos y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + t\left(\frac{-1}{1+x}\right) = (1+x)e^{x}$$

$P(x) = \frac{-1}{1+x}$ $\quad Q(x) = (1+x)e^{x}$

$$I_F = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{-1}{1+x}dx}$$

Let $1+x = a$

$$e^{-\log|x+1|}$$

$$e^{\log(x+1)^{-1}}$$

$$I_F = (x+1)^{-1}$$

$$t \cdot \frac{1}{x+1} = \int \frac{1}{\cancel{x+1}} \cancel{(1+x)} e^x dx + c$$

$$\frac{t}{x+1} = \int e^x dx + c$$

$$\therefore \frac{\sin y}{x+1} = e^x + c$$ is the general

solution of Equation $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$.

5 01.05.2023

Orthogonal Trajectories (only in cartesian coordinates)

A curve which cuts every member of given family of curves ~~of~~ at right angle is known as orthogonal trajectories of given family of curves.

- Working Rule :

- The family orthogonal trajectries must be in cartesian form.

$$F(x, y, c) = 0 \quad — (1)$$

· Eleminati constants in Eq ① by derivation w.r.t x.

$$\Rightarrow F\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{—— ②}$$

· Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq ②

i.e, $$F\left(x, y, \frac{-dx}{dy}\right) = 0 \quad \text{—— ③}$$

· Solve Eq ③ then we get equation of family of orthogonal trajectories.

Q Find the orthogonal trajectry of the family of $y = ax$.

Solution:-

$$y = ax \quad \text{— ①}$$

differenciation Eq ① wrt 'x'

$$\frac{d}{dx} y = \frac{d}{dx}(ax)$$

$$\frac{dy}{dx} = a \cancel{\frac{dx}{dy}} \cancel{(2)} \frac{dx}{dx}$$

$$\frac{dy}{dx} = a$$

① $\Rightarrow$ $$y = \frac{dy}{dx} x \quad \text{—— ②}$$

The differential Equation of family of curves:

$$y = x \frac{dy}{dx}$$

Replace $\left[\frac{dy}{dx} = \frac{-dx}{dy} \right]$

$$y = x\left(\frac{-dx}{dy}\right) \quad — (3)$$

$$y\,dy = -x\,dx$$

Integrating on both sides

$$\int y\,dy = \int -x\,dx + c$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + c$$

$\therefore \quad \frac{y^2}{2} + \frac{x^2}{2} = c'$ is the orthogonal trajectories

Q Find the orthogonal Trajectory of family of curves $x^2 + y^2 = a^2$.

Solution:

$$x^2 + y^2 = a^2 \quad — (1)$$

Differencicating Eq(1) wrt x.

$$\frac{d}{dx}(x^2 + y^2) = \frac{da^2}{dx}$$

$$2x + 2y\frac{dy}{dx} = 0.$$

$$x + y\frac{dy}{dx} = 0 \quad — (2)$$

The D.E of family of curves!

$$x + y\frac{dy}{dx} = 0$$

Replace $\frac{dy}{dx}$ by $\frac{-dx}{dy} = 0$

$$x + y\left(\frac{-dx}{dy}\right) = 0$$

$$x\,dy - y\,dx = 0.$$

$$x\,dy = y\,dx$$

$$\frac{1}{x}dx = \frac{1}{y}dy$$

Integrating on both sides

$$\int \frac{1}{x}dx = \int \frac{1}{y}dy + c$$

$$\log x = \log y + \log c$$

$$\log x/y = \log c$$

$$\frac{x}{y} = c$$

$x = cy$ is orthogonal trajectory.

Q Find the orthogonal trajectories of family of semi cubical parabola $ay^2 = x^3$.

Sol:

$$ay^2 = x^3 \quad \text{—} \; (1)$$

Derivative Eq (1) w.r.t x,

$$\frac{d}{dx}ay^2 = \frac{d}{dx}x^3$$

$$2ay\frac{dy}{dx} = \cancel{\frac{d}{dx}x^3}\quad 3x^2 \quad —(2)$$

$$(1) \Rightarrow \frac{2\cancel{a}y\cdot\frac{dy}{dx}}{\cancel{a}y^{\cancel{2}}} = \frac{3\cancel{x^2}}{x^{\cancel{3}}}$$

$$\frac{2}{y}\frac{dy}{dx} - \frac{3}{x} = 0.$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$\frac{2}{y} - \frac{dx}{dy} = \frac{3}{x}.$$

$$2x(-dx) = 3y\,d\cdot y + c$$

$$-2\int x\,dx = 3\int y\,dy + c$$

$$-\cancel{2}\frac{x^2}{\cancel{2}} = \frac{3y^2}{2} + c$$

$\therefore$ $x^2 + \frac{3y^2}{2} + c = 0$ is orthogonal trajectory.

02.05.2023.

1) DO that the system of Parabola $y^2 = 4a(x+a)$ is self orthogonal.

Sol:- Given,

family of circle.

$$y^2 = 4a(x+a) \rightarrow (1)$$

Derivative wrt 'x'.

$$2y\frac{dy}{dx} = 4a(1+0).$$

$$2yy' = 4a$$

$$a = \frac{yy'}{2}.$$

Substitute 'a' in eq (1).

$$y^2 = \frac{\cancel{4}(yy')}{\cancel{2}}\left(x + \frac{yy'}{2}\right)$$

$$y = \cancel{2}y'\left(\frac{2x+yy'}{\cancel{2}}\right)$$

$$\boxed{y = 2xy' + y(y')^2} \rightarrow (2).$$

The D.E of family of circle

Replace y' to $-1/y'$ in eq (2).

$$y = 2x\left(\frac{-1}{y'}\right) + y\left(\frac{-1}{y'}\right)^2$$

$$y = \frac{-2x}{y'} + y\left(\frac{1}{y'}\right)^2.$$

$$y = \frac{-2xy' + y}{y'^2}$$

$$yy'^2 = -2xy' + y$$

$$\rightarrow \boxed{y = y(y')^2 + 2xy'} \rightarrow (3)$$

then eq (2) & (3) are same so, eq (1) is self orthogonal.

H/W

(1). S/T that system of $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where λ is parameter is self-orthogonal.

---

(6) Imp(*). Newton's law of cooling:-

Statement - I

The rate of change of Temperature of a body is directly propotional to the difference b/w temperature of the body that the surrounding medium.

$$\boxed{\frac{d\theta}{dt} \propto (\theta - \theta_0)}$$

θ is the temperature of the body at time T.

and θ_0 be the temperature of surrounding medium - (or) room temperature.

1) A body is orginally 80°c and cools down to 60°c in 20 min. If the temperature of air is 40°c find the temperature of the body after 40 min.

sol:- step(1):-

$t=0, \theta=80$

$t=20, \theta=60$

$\theta_0=40$

$t=40, \theta=?$

step(2):- $(\theta-\theta_0)=ce^{-Kt} \rightarrow (1)$.

step(3):- find c, value.

$\theta-\theta_0=ce^{-Kt}$

$t=0, \theta=80$

$(80-40)=ce^{-K(0)}$

$\boxed{40=c}$

step(4):- find (K value).

$t=20, \theta=60.$

$(60-40)=40e^{-K(20)}$

$2\cancel{0}=4\cancel{0}e^{-20K}$

$\frac{1}{2}=e^{-20K}$

$\log\frac{1}{2}=\log e^{-20K}$

$\log \frac{1}{2} = -20K.$

$K = -\frac{1}{20} \log \frac{1}{2}$

$\boxed{K = 0.035}$

Step(5):- substitute c and k in eq ①.

$(\theta - \theta_0) = 40 e^{(-0.035)t}$

$(\theta - 40) = 40 e^{(-0.035)40}.$

$\theta = 40 + 40 e^{(-0.035)40}.$

$\boxed{\theta = 49.86°c}.$

---

2) A body kept in a air with Temperature 25° cools from 140° to 80°c in 20 min find when the body cools down 35°c.

Sol:- Step(1):-

$t = 0, \quad \theta = 140°c$

$t = 20 min, \quad \theta = 80°c.$

$\theta_0 = 25$

$t = ? \quad \theta = 35°c.$

Step(2):-

$(\theta - \theta_0) = c e^{-kt} \longrightarrow ①.$

Step(3):- $t = 0, \theta = 140°c$.

$$\theta - \theta_0 = ce^{-kt}$$

$$(140 - 25) = ce^{-k(0)}$$

$$\boxed{115 = c}$$

Step(4):- $\theta = 80°c, t = 20$.

$$\theta = \theta_0 = ce^{-kt}$$

$$(80 - 25) = 115e^{-k(20)}.$$

$$55 = 115e^{-20k}$$

$$\frac{55}{115} = e^{-20k}$$

$$\log\frac{55}{115} = \log e^{-20k}$$

$$-20k = \log\frac{55}{115}$$

$$k = \frac{-1}{20} \ln\frac{55}{115}$$

$$\boxed{k = 0.036}$$

Step(5):- when $\theta = 35°c$ $t = ?$

$$(\theta - \theta_0) = ce^{-kt}$$

$$(35 - 25) = 115e^{-(0.036)t}$$

$$\frac{10}{115} = e^{-0.036t}.$$

$$\log\frac{10}{115} = \log e^{-0.036t}$$

$-0.036t = \ln 10/115$

$t = \frac{-1}{0.036} \ln\left(\frac{10}{115}\right)$

$\boxed{t = 67 \text{ min}}$

Q If the air is maintained at $20°$ and the temp of the body drops from $100°$ to $80°$ in 10 min. What will be the temp after 20 min, when will be the temp $40°c$.

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Q

$\theta_0 = 20°$

$\boxed{\theta - \theta_0 = c e^{-kt}}$

$t = 0 \to \theta = 100°$

$t = 10 \to \theta = 80°$

$t = 20 \to \theta = ?$

$t = ? \to \theta = 40°.$

(i) $t = 0, \quad \theta = 100°$

$100 - 20 = c e^{-k(0)}$

$80 = c e^{0}$

$\boxed{c = 80.}$

(ii) $t = 10, \quad \theta = 80.$

$80 - 20 = 80 e^{-k(10)}$

$k = \log\left(\frac{60}{80}\right) \times \left(\frac{-1}{10}\right).$

$\boxed{k = 0.0287.}$

iii $t = 20$

$\theta - 20 = 80 e^{-(0.0287)(20)}$

$\theta = 20 + 80 \times e^{-0.0287(20)}$

$\theta = 65.061°.$

$\theta = 40$

$$40 - 20 = 80\, e^{-(0.0287)(t)}$$

$$t = \frac{-1}{0.0287} \times \log_e\left(\frac{20}{80}\right) = 48 \text{ min}$$

$\therefore$ after 20 min, temperature will be 65° and It will be 48 min to be 40°.

Formula derivation -

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

Proportionality Constant = $-k$.

$$\frac{d\theta}{dt} = (-k)(\theta - \theta_0)$$

$$\int \frac{1}{\theta - \theta_0}\, d\theta = (-k)\int dt + \log c$$

Constant = $\log c$

$$\log\left(\frac{\theta - \theta_0}{c}\right) = -kt.$$

$$\boxed{\theta - \theta_0 = c e^{-kt}}$$

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9 If the air is maintained at $30^\circ C$ and temp of the body cools from $80^\circ C$ to $60^\circ C$. find the temp after 24 min. and when temp will be at $40^\circ C$

Solution :

$$\theta_0 = 30^\circ c$$

$t = 0$ $\theta = 80$

$t = 12$ $\theta = 60^\circ$

$t = 24$ $\to \theta = ?$

$t = ?$ $\theta = 40^\circ c$

$$\theta - \theta_c = c\, e^{-kt}$$

(i) $t = 0, \theta = 80$

$$80 - 30 = c\, e^{-k(0)}$$

$$c = 50$$

(ii) $t = 12, \theta = 60$

$$60 - 30 = 50\, e^{-k(12)}$$

$$k = \frac{-1}{12} \times \log\left(\frac{30}{50}\right)$$

$$k = 0.04256$$

(iii) $t = 24$

$$\theta - 30 = 50\, e^{-(0.04256)(24)}$$

$$\theta = 30 +$$

$$\theta = 48^\circ c$$

$$40-30 = 50\, e^{-0.04256\,(t)}$$

$$t = \frac{-1}{0.04256} \times \log\left(\frac{10}{50}\right)$$

$$t = 37.81 \text{ min}.$$

$\therefore$ after 24 min, the temp will be $48°C$ and at 37.81 min, tem will be $40°C$.

Law of Natural Growth and Decay:-

Let X be the amount of substance available at the time t. A low of chemical conversation states that the rate of change of amount of chemically changing substance is proportional to the amount of substance available at that time.

$$\boxed{\frac{dX}{dt} \propto X}$$

$$\frac{dX}{dt} = \pm k X$$

$$\frac{1}{X}\, dx = \pm k\, dt$$

$$\log X = \pm kt + \log c$$

$$\log\left(\frac{X}{c}\right) = \pm kt$$

$$\boxed{X = c\, e^{\pm kt}}$$

Growth: $X = ce^{kt}$

Decay: $X = ce^{-kt}$

Q Bacteria is growing exponenctially increses from 200g to 500g in the period from 6AM to 9AM (3hrs). How many gram present at noon. (6hrs).

$t = 0\,hr \quad X = 200g$

$t = 3\,hr \quad X = 500g$

$t = 6\,hrs \quad X = ?$

$X = c\,e^{kt}$

Case I :- $X = 200g, \quad t = 0\,hrs$

$$200 = c\,e^{k(0)}$$

$$200 = c\,e^{0}$$

$$c = 200$$

Case II : $X = 500g, \; t = 3\,hrs$

$$500 = (200)\,e^{k(3)}$$

$$k = \frac{1}{3} \times \log_e\left(\frac{500}{200}\right)$$

$$k = 0.30543$$

Case III: $t = 6$ hrs

$$X = (200)\, e^{(0.30543)(6)}$$

$$X = 1249.99\ g$$

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Case IV: $N = 100$ $t = 0$

$N = 400$ $t = 10$

$N = ?$ $t = 3$

I :- $X = 100$ grms, $t = 0$

$$100 = c\, e^{k(0)}$$

$$c = 100$$

II :- $X = 400$ grms, $t = 10$

$$400 = (100)\, e^{k(10)}$$

$$k = \frac{1}{10}\left(\log\left(\frac{1}{4}\right)\right)$$

$$k = 0.1386$$

III :- $X = ?$ $t = 3$

$$X = (100)\, e^{(0.1386)(3)}$$

$X = 151.55$ grams after 3 hrs.

Q. The rate at which bactoria multiplies is proportional to the instantenous N present. If the original N present doubles in 2 hrs when will it be triple.

$$x = N \quad t = 0$$

$$x = 2N \quad t = 2 \text{ hrs}$$

$$x = 3N \quad t = ?$$

(i) :- $$x = c e^{kt}$$

$x = N, t = 0$

$$N = c e^{k(0)}$$

$$C = N$$

(ii) :- $x = 2N, t = 2$

$$2N = (N) e^{k(2)}$$

$$k = \frac{1}{2} \log(2).$$

$$k = 0.34657$$

(iii) :- $x = 3N, t = ?$

$$3N = (N) e^{(0.34657)(t)}$$

$$t = \frac{1}{0.34657} \log(3.)$$

$$= 3.1699 \text{ hrs} = 3 \text{ hrs } 10 \text{ min.}$$

The number N of bacteria is grown at rate proportional ~~A~~ to N. The value of N was initially 100 and increased 332 in 1hrs. what was the value of N after 1.5hrs.

$N = 100 \qquad t = 0$

$N = 332 \qquad t = 1 = 60$ min

$N = ? \qquad t = 3/2 = 90$ min

$$x = c e^{kt}$$

I $\quad x = 100, t = 0$

$$100 = c e^{k(0)}$$

$$c = 100$$

II $\quad x = 332 \quad t = 60$

$$332 = 100 (e)^{k(60)}$$

$$k = \frac{1}{60} \log \left(\frac{332}{100} \right)$$

$$k = 0.019999$$

III $\quad x = ? \quad T = 90$

$$x = 100\, e^{(0.01999)(90)}$$

~~x = 331 min~~

~~= 5.5 hrs~~

~~= 5 hrs 31 min.~~

$$x = 604.91.$$